Probability Review

Why Probabilistic Robotics?

- autonomous mobile robots need to accommodate the uncertainty that exists in the physical world
- sources of uncertainty
  - environment
  - sensors
  - actuation
  - software
  - algorithmic
- probabilistic robotics attempts to represent uncertainty using the calculus of probability theory

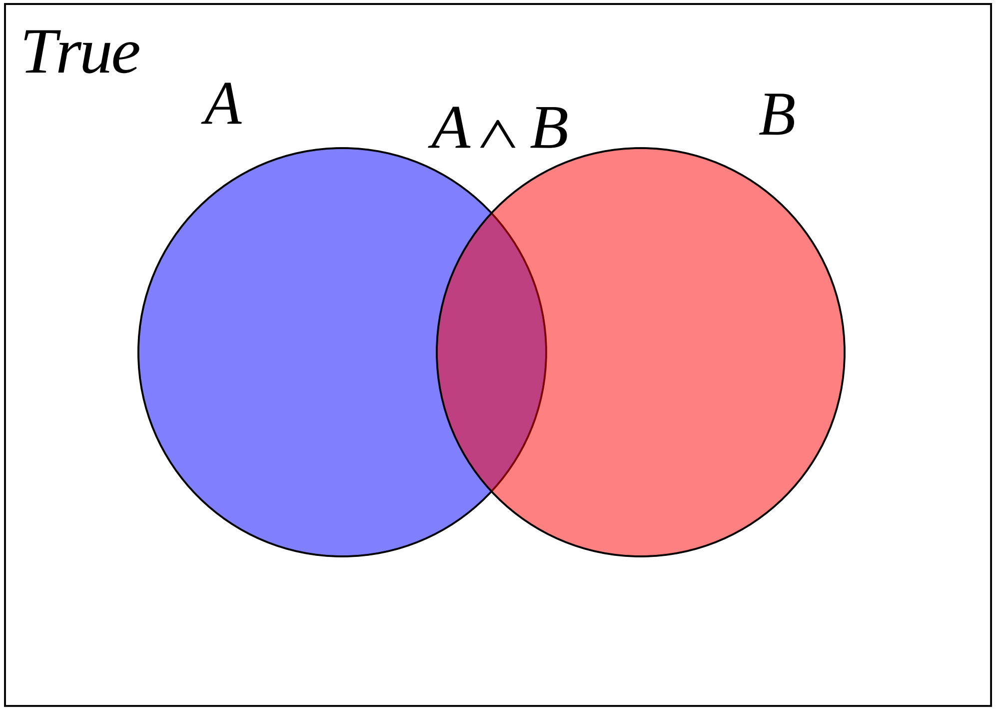
Axioms of Probability Theory

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \le \Pr(A) \le 1$
- $\Pr(True) = 1$ $\Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

$$\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}$$

Discrete Random Variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\cdot)$ is called probability mass function.

Discrete Random Variables

- fair coin

$$P(X=heads) = P(X=tails) = 1/2$$

- fair dice

$$P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6) = 1/6$$

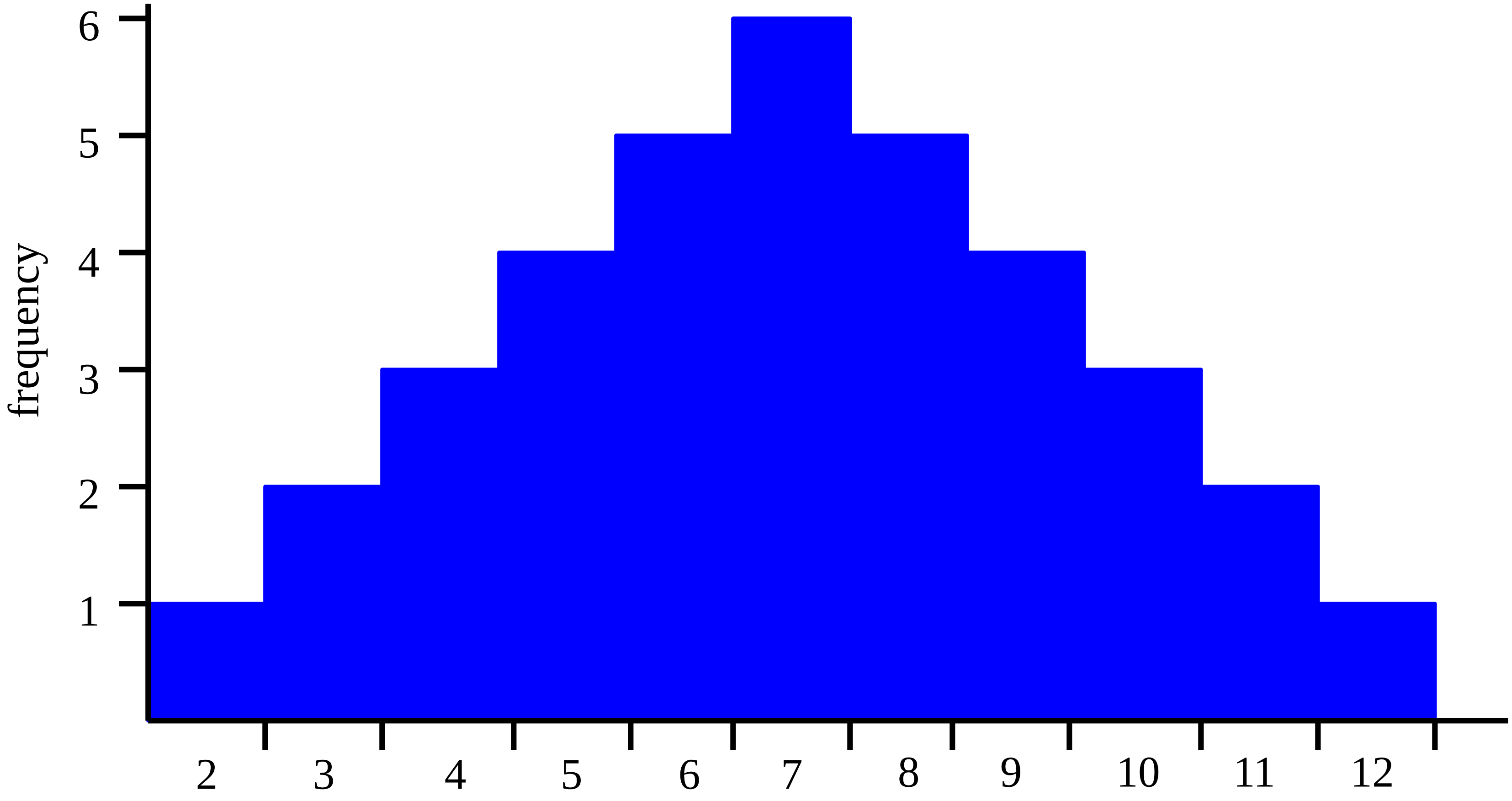
Discrete Random Variables

- sum of two fair dice

P(X=2)	(1,1)	1/36
P(X=3)	(1,2), (2,3)	2/36
P(X=4)	(1,3), (2,2), (3,1)	3/36
P(X=5)	(1,4), (2,3), (3,2), (4,1)	4/36
P(X=6)	(1,5), (2,4), (3,3), (4,2), (5,1)	5/36
P(X=7)	(1,6), (2,5), (3,4), (4,3), (5,2), (6, 1)	6/36
P(X=8)	(2, 6), (3, 5), (4,4), (5,3), (6, 2)	5/36
P(X=9)	(3, 6), (4, 5), (5, 4), (6, 3)	4/36
P(X=10)	(4, 6), (5, 5), (6, 4)	3/36
P(X=11)	(5, 6), (6, 5)	2/36
P(X=12)	(6, 6)	1/36

Discrete Random Variables

- plotting the frequency of each possible value yields the histogram

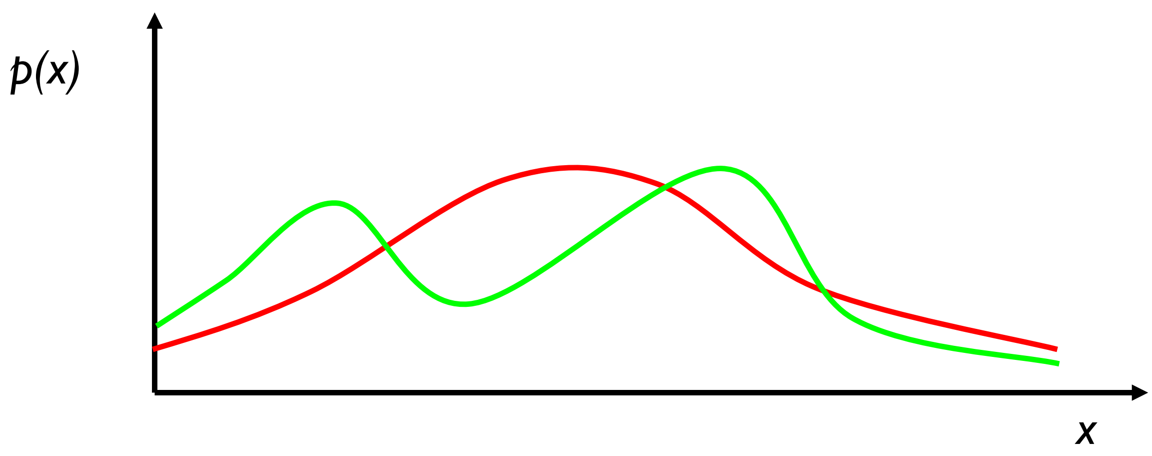


Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

- E.g.



Continuous Random Variables

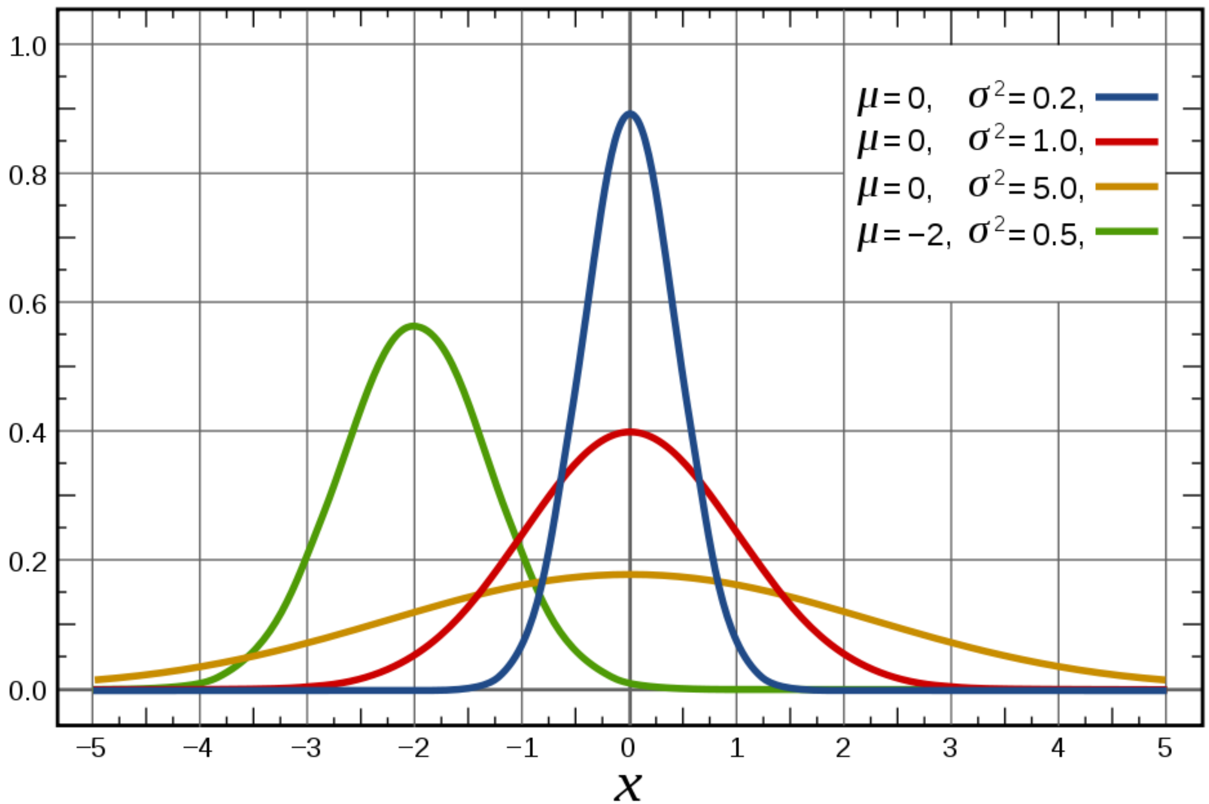
- unlike probabilities and probability mass functions, a probability density function can take on values greater than 1
  - e.g., uniform distribution over the range [0, 0.1]
- however, it is the case that

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Continuous Random Variables

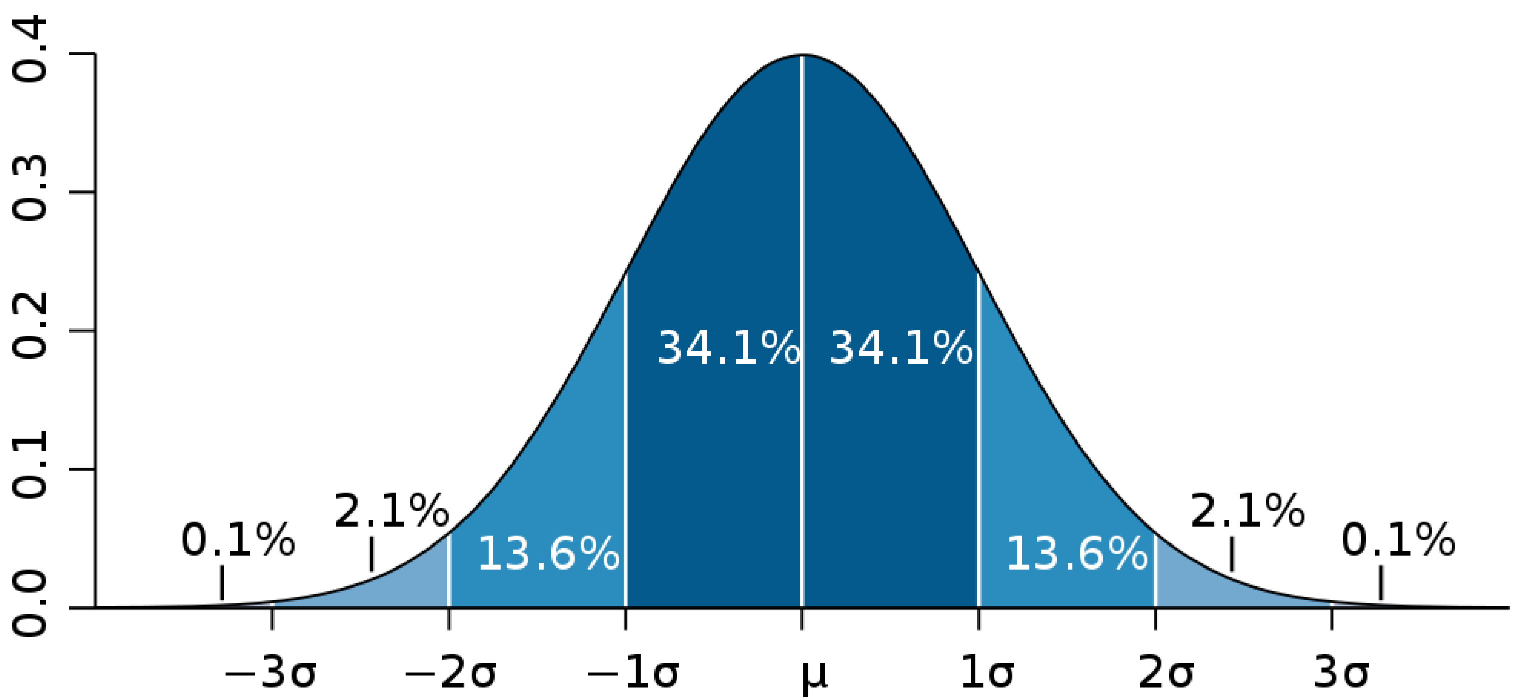
- normal or Gaussian distribution in 1D

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Continuous Random Variables

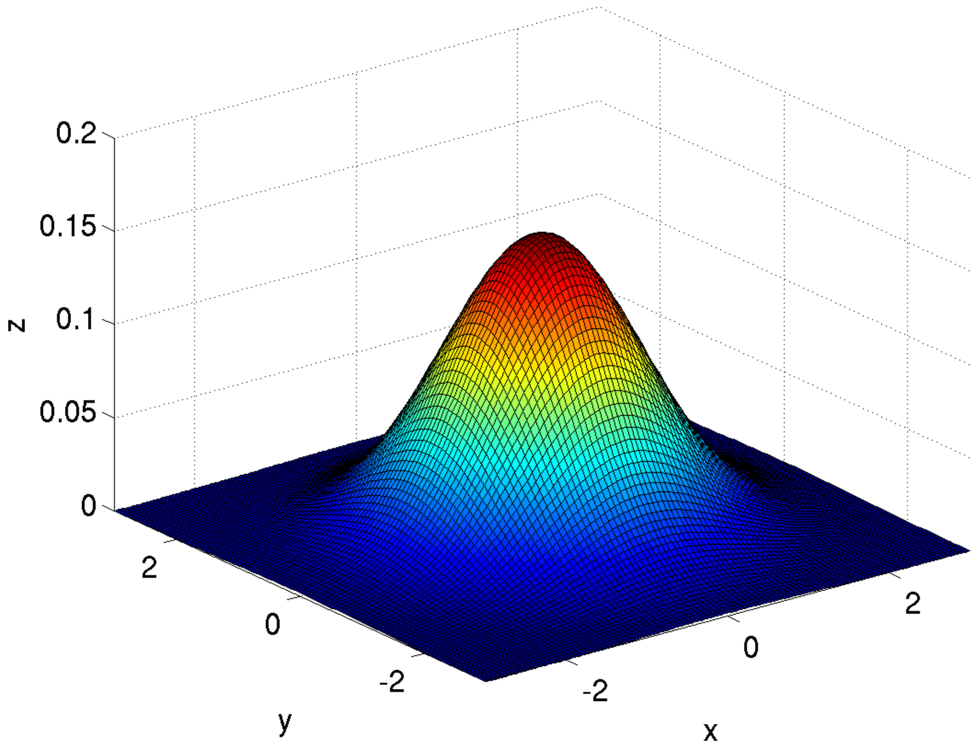
- $1D$ normal, or Gaussian, distribution
  - μ mean
  - σ standard deviation
  - $\Sigma = \sigma^2$ variance



Continuous Random Variables

- $2D$ normal, or Gaussian, distribution
  - μ mean
  - Σ covariance matrix

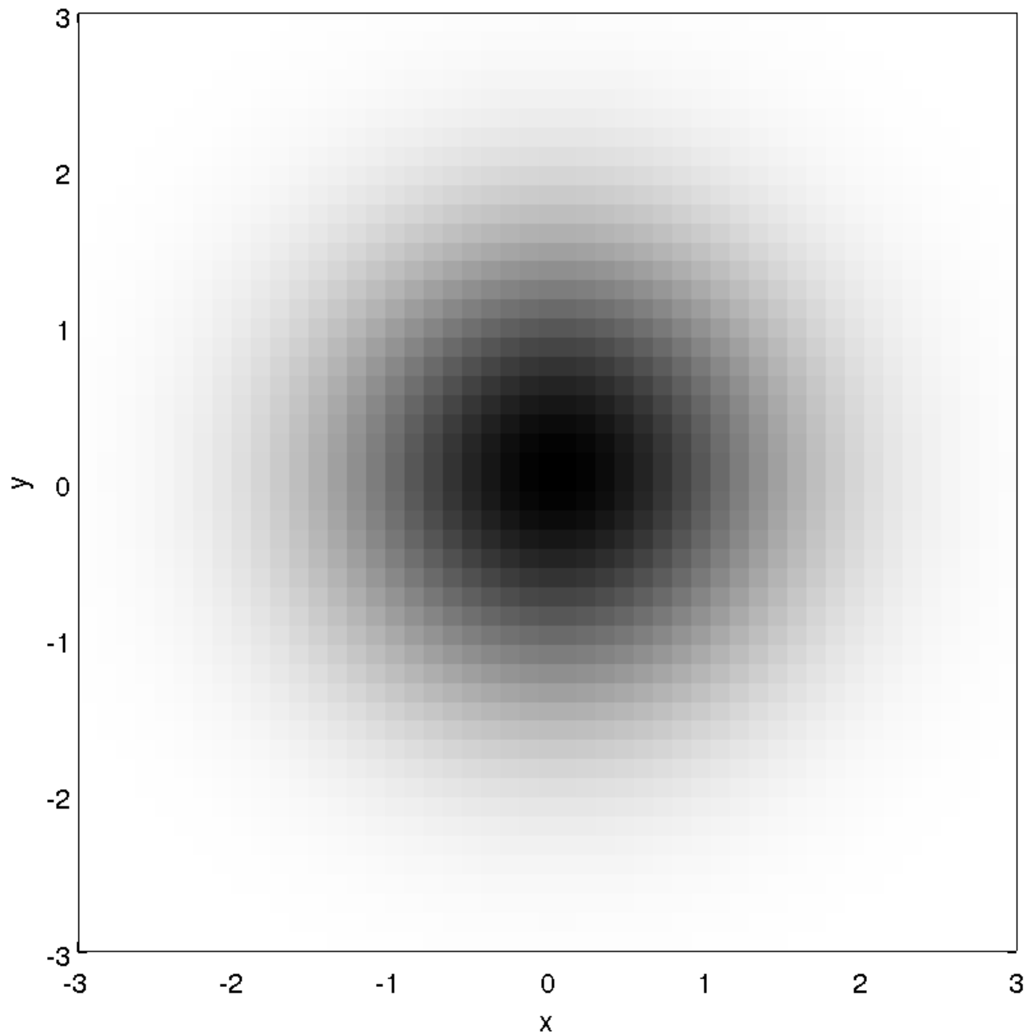
$$p(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$



Continuous Random Variables

- in $2D$
  - isotropic

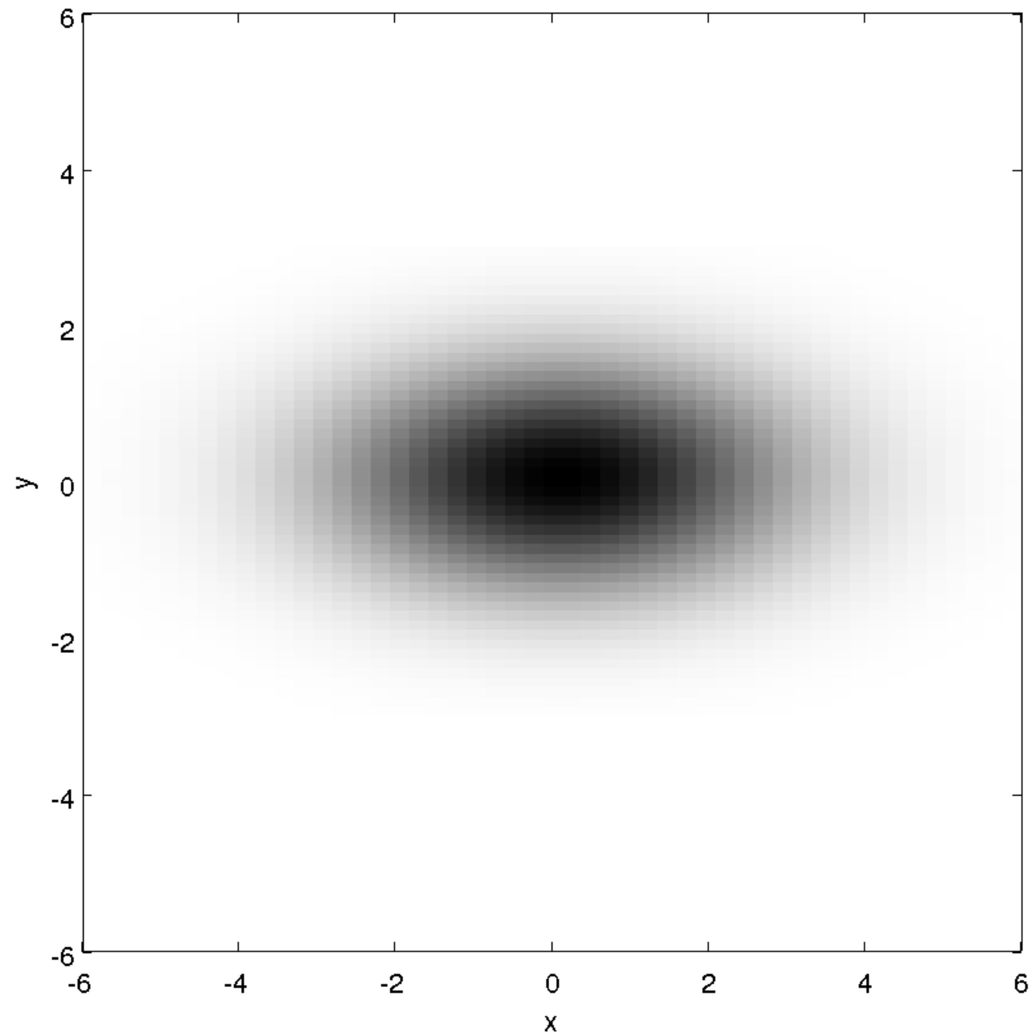
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

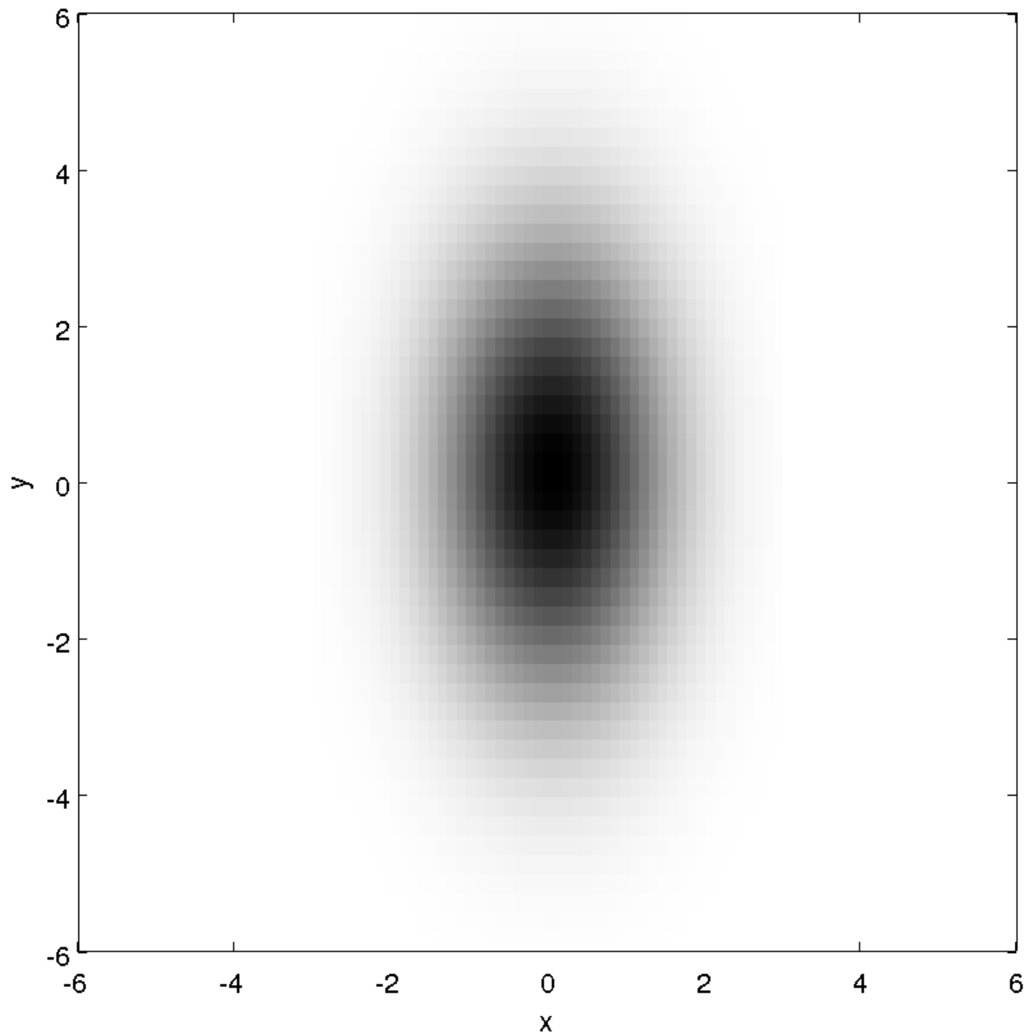
$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

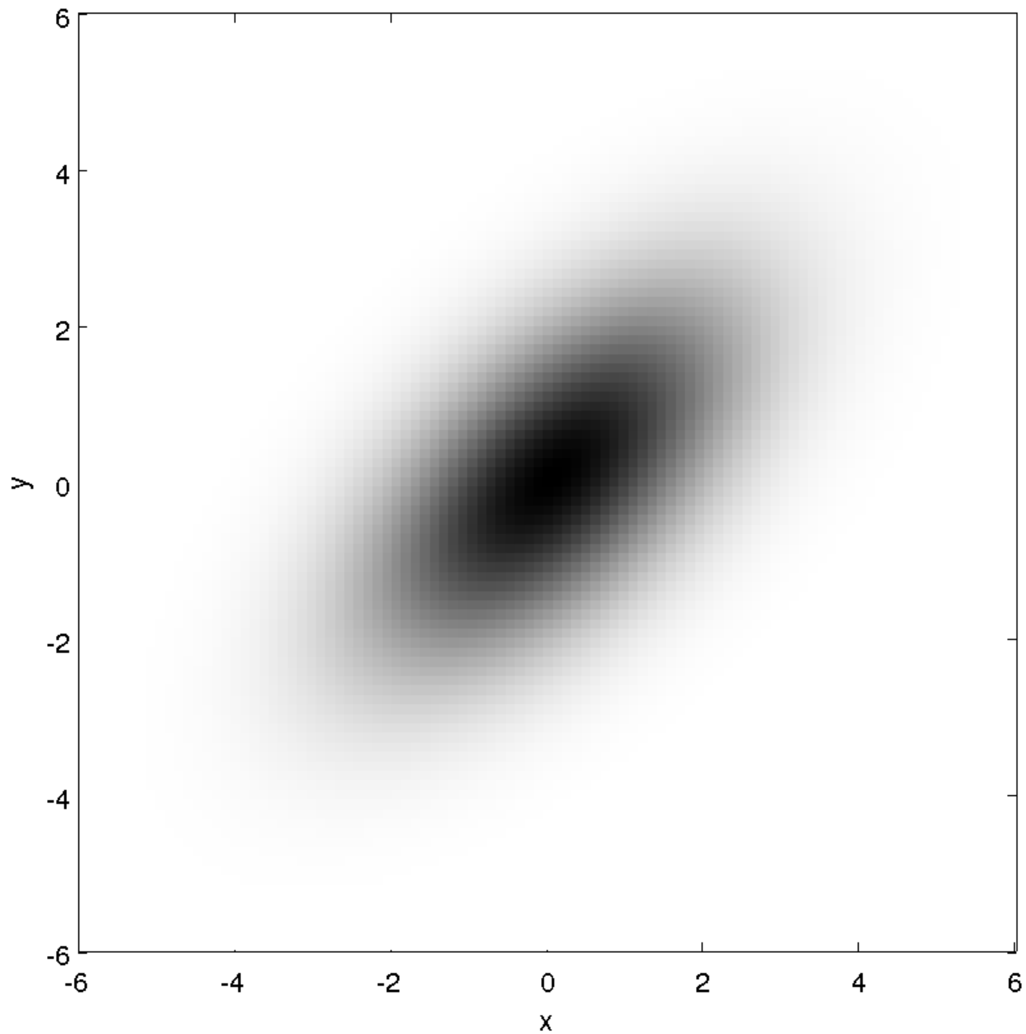
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$



Continuous Random Variables

- in $2D$
  - anisotropic

$$\Sigma = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$$



Covariance matrices

- the covariance matrix is always symmetric and positive semi-definite
- positive semi-definite:

$$x^T \Sigma x \geq 0 \text{ for all } x$$

- positive semi-definiteness guarantees that the eigenvalues of Σ are all greater than or equal to 0

```
>> [v, d] = eig([1 0; 0 4])

v =

     1     0
     0     1


d =

     1     0
     0     4
```

```
>> [v, d] = eig([2.5 1.5; 1.5 2.5])

v =

   -0.7071    0.7071
    0.7071    0.7071


d =

     1     0
     0     4
```

Joint Probability

- the joint probability distribution of two random variables

$$P(X=x \text{ and } Y=y) = P(x,y)$$

  describes the probability of the event that X has the value x and Y has the value y

- If X and Y are independent then

$$P(x,y) = P(x)\,P(y)$$

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- example: two fair dice

$$P(X{=}\text{even and } Y{=}\text{even}) = 9/36$$

$$P(X{=}1 \text{ and } Y{=}\text{not } 1) = 5/36$$

Joint Probability

- example: insurance policy deductibles

y

x		$0	$100	$200
	$100	0.20	0.10	0.20
	$250	0.05	0.15	0.30

home

automobile

Joint Probability and Independence

- X and Y are said to be independent if

$$P(x,y) = P(x)\,P(y)$$

  for all possible values of x and y

- example: two fair dice

$$P(X\text{=even and }Y\text{=even}) = (1/2)\,(1/2)$$
$$P(X\text{=1 and }Y\text{=not 1}) = (1/6)\,(5/6)$$

- are X and Y independent in the insurance deductible example?